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Rapports de Recherche

N° 365

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Février 1985

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ABSTRACT :

This paper is devoted to the two-product problem with joint production costs and independent inventory costs. It gives an algorithm which leads to the optimal policy in the infinite horizon case. This algorithm is the extension of the celebrated EOQ formula which was obtained for the mono-product problem.

RESUME :

Ce papier est consacré au problème à deux produits avec incitation aux lançements groupés. Il propose un algorithme qui donne la politique optimale lorsque l'horizon est infini. Cet algorithme prolonge la célèbre formule des quantités économiques établie dans le cas d'un produit unique.

1. INTRODUCTION

Many papers have been devoted to the multi-products problem with concave, individual inventory costs and concave, joint production costs (see in particular A.EDWARD SILVER (1979), P.C. EDWARD KAO (1979), V.I.LEOPOULOS and J.M.PROTH (1984)). On the other hand, the optimal policy of the mono-product problem with concave costs, constant demand and infinite horizon is given by Wilson's formula. In the following, we propose two algorithms which lead to the optimal control of the two-products problem with individual inventory costs and joint production costs when the horizon is infinite. The demand is constant at each time unit for every product and the initial inventory levels are equal to zero. We found that the optimal policy is periodic. The algorithms lead to the restriction of the optimal control to one period.

The paper is organized as follows:

- we first give some properties of a particular mono-product problem.
- we give some results concerning the two-products problem and deduce an algorithm which leads to the E.O.Q.
- we finally bring up an exemple.

2. SOME PROPERTIES OF A PARTICULAR MONO-PRODUCT PROBLEM

2.1. NOTATION

Let N be the horizon of the problem. $\xi_i \geq 0$ is the demand at time i $(i=1,2,\dots,N)$. $\nu_i \geq 0$ is the production level (or replenishment) decided at time i and available at time i+1 $(i=0,1,\dots,N-1)$. $V=\{\nu_0,\nu_1,\dots,\nu_{N-1}\}$ is called a control (or solution) of the problem. We denote by y_i $(i=0,1,\dots,N)$ the inventory level on $[i,i+1)$. The following relations are called state equations:

$$y_{i+1}= y_i + \nu_i - \xi_i, \quad i= 0, 1,\dots, N-1 \tag{2-1}$$

We suppose that backlogging is not allowed, i.e. :

$$y_i \geq 0 \text{ for } i= 0, 1,\dots, N \tag{2-2}$$

$Y=(y_0,y_1,\dots,y_N)$ is the sequence of inventory levels corresponding to V starting from y_0 and using (2-1).

We also consider two types of cost functions for $i=0, 1,\dots, N-1$:

1. $f_i(y)=by$, $b\in\mathbb{R}^{*+}$ and $y\in\mathbb{R}^+$, is the cost of holding in stock a quantity y at time i for one period until i+1. f_i is positive, nom decreasing and concave.

2. $c_i(\nu)= K_i X_{\nu>0}+ a\nu$, where $X_{\nu>0}= 1$ if $\nu\in\mathbb{R}^+$ and 0 otherwise, $K_i\in\mathbb{R}^+$ and $a\in\mathbb{R}^+$, is the cost of ordering a quantity ν at time i. c_i is positive, non decreasing and concave.

Let us consider a control V and the corresponding sequence of inventory levels. If Y verifies (2-2), V is said to be admissible.

The total cost corresponding to the admissible control V depends on y_0. It is denoted by $\mathscr{C}(y_0,V)$:

$$\mathscr{C}(y_0,V) = \sum_{i=0}^{N-1} [c_i(\nu_i) + f_i(y_i)]$$

or :

$$\mathscr{C}(y_0,V) = b\sum_{i=0}^{N-1} y_i + \sum_{i=0}^{N-1} X_{\nu_i>0}K_i + a\sum_{i=0}^{N-1} \nu_i \tag{2-3}$$

Let D be the set of admissible controls. If $V^* \in D$ and

$$\mathscr{C}(y_0,V^*) = \underset{V\in D}{\mathrm{Min}}\,\mathscr{C}(y_0,V) \;,$$

then V^* is said to be optimal for the N-horizon problem.

Note that K_i depends on i :the above problem is then non stationary and more general than Wilson's one.

2.2. PROPERTIES OF THE OPTIMAL CONTROL

The following results are well known (see in particular [1], [2], [3] and [4]). They hold for every problem with concave and non decreasing cost functions.

There exists an optimal control $V=\{\nu_0, \nu_1,\ldots, \nu_{N-1}\}$ which verifies :

1. $\nu_i > 0$ if and only if $y_i<\xi$, except perhaps for the first strictly positive ν_i ($Y=\{y_0,\ldots, y_N\}$ is the sequence of inventory levels corresponding to V).
2. if $\nu_i > 0$, then $y_i+\nu_i = (r-i)\xi$, with $r \in \{i+1,\ldots, N\}$ (2-4)

In the case of $K_i = K\in\mathbb{R}^+$, $\forall i \in \{0, 1,\ldots, N-1\}$, the problem becomes stationary and property 1 is true without exception.

2.3. ADDITIONAL PROPERTIES OF THE PROBLEM DEFINED IN 2.1.

Let $V=\{\nu_0, \nu_1,\ldots, \nu_{N-1}\}$ be an optimal control which verifies (2-4). We denote by P_0 the above problem and by P_1 the problem obtained starting from P_0 by putting $K^1_{i_1}$ in the place of K_{i_1} for an $i_1 \in \{0, 1,\ldots, N-1\}$ which satisfies $\nu_{i_1} > 0$. We choose $K^1_{i_1} < K_{i_1}$.

Theorem (2-1)

V is also optimal for P_1.

Proof :

Let us consider an admissible control W which is not optimal for P_1 :

$$\mathcal{C}^0(y_0,V) < \mathcal{C}^0(y_0,W) \tag{2-5}$$

Furthermore, considering the definition of P_1 and (2-5) :

$$\mathcal{C}^1(y_0,V) = \mathcal{C}^0(y_0,V) - (K_{i_1} - K^1_{i_1}) < \mathcal{C}^0(y_0,W) - (K_{i_1} - K^1_{i_1}) \tag{2-6}$$

If $W = \{w_0, w_1, \ldots, w_{N-1}\}$, we have to consider two cases :

1. $w_{i_1} = 0$. In that case :

$$\mathcal{C}^0(y_0,W) = \mathcal{C}^1(y_0,W) \tag{2-7}$$

2. $w_{i_1} > 0$. In that case :

$$\mathcal{C}^0(y_0,W) = \mathcal{C}^1(y_0,W) + (K_{i_1} - K^1_{i_1}) \tag{2-8}$$

From (2-7) and (2-8) :

$$\mathcal{C}^0(y_0,W) - (K_{i_1} - K^1_{i_1}) \leqq \mathcal{C}^1(y_0,W) \tag{2-9}$$

Finally, (2-6) and (2-9) lead to :

$$\mathcal{C}^1(y_0,V) < \mathcal{C}^1(y_0,W)$$

□

Remarks :

1. If $\nu_{i_1} = 0$, theorem (2-1) doesn't hold.

2. If V is optimal for P_1, $\nu_{i_1} = 0$ and $K^1_{i_1} < K_{i_1}$, then it is possible to prove similarly as above that V is optimal for P.

Theorem (2-2)

Let $V^0 = \{\nu_0^0, \nu_1^0, \ldots, \nu_{N-1}^0\}$ an optimal control for P_0 and $V^1 = \{\nu_0^1, \nu_1^1, \ldots, \nu_{N-1}^1\}$ an optimal control for P_1.

If $\nu_{i_1}^0 = \nu_{i_1}^1 = 0$, then V^0 is optimal for P_1 and V^1 is optimal for P_0.

Proof :

$\nu_{i_1}^0 = 0$ leads to :

$$\mathscr{C}^1(y_0, V^0) = \mathscr{C}^0(y_0, V^0) \tag{2-10}$$

V^0 is optimal for P_0, then :

$$\mathscr{C}^0(y_0, V^0) \leq \mathscr{C}^0(y_0, V^1) \tag{2-11}$$

$\nu_{i_1}^1$ being equal to zero, we can write :

$$\mathscr{C}^0(y_0, V^1) = \mathscr{C}^1(y_0, V^1) \tag{2-12}$$

We deduce from (2-10), (2-11) and (2-12) :

$$\mathscr{C}^1(y_0, V^0) \leq \mathscr{C}^1(y_0, V^1) \tag{2-13}$$

And, V^1 being optimal for P_1, we can write :

$$\mathscr{C}^1(y_0, V^1) \leq \mathscr{C}^1(y_0, V^0) \tag{2-14}$$

Finally, considering (2-13) and (2-14) :

$$\mathscr{C}^1(y_0, V^0) = \mathscr{C}^1(y_0, V^1) \tag{2-15}$$

and V^0 is optimal for P_1.

Considering (2-10), (2-12) and (2-15) we obtain :

$$\mathscr{C}^0(y_0, V^0) = \mathscr{C}^0(y_0, V^1)$$

and V^1 is optimal for P_0. □

2.4. THE OPTIMAL CONTROL OF THE N-HORIZON STATIONARY PROBLEM

Let us now consider the N-horizon problem defined in 2.1 with $K_i=K$ whatever $i \in \{0,1,\dots,N-1\}$ may be. This problem is stationary. In addition, we assume that $y_0=0$. We denote by $\mathcal{E}(x)$ the integer part of every $x \in \mathbb{R}$.

The following properties will be useful in the study of the two-products problem.

Theorem (2-3)

Let be $p=\left(-1+\sqrt{1+4\frac{2K}{b\xi}}\right)/2.$ (2-16)

p is always greater than zero.

1. If $p > N$, the components of the optimal control are equal to zero, except the first one which is equal to $N\xi$.

2. If $p \leq N$, we set :

$$a=\begin{cases} N/\mathcal{E}(p) & \text{if } \mathcal{E}(p)=p \\ N/(\mathcal{E}(p)+1) & \text{if } \mathcal{E}(p)\neq p \end{cases} \tag{2-17}$$

and ;

$$q_1=\mathcal{E}(N/\mathcal{E}(a)),\ s_1=N-q_1\,.\mathcal{E}(a) \tag{2-18}$$

$$q_2=\mathcal{E}(N/(\mathcal{E}(a)+1)),\ s_2=N-q_2(\mathcal{E}(a)+1) \tag{2-19}$$

2.1. Let us first assume that $a=\mathcal{E}(a)$. In that case, an admissible control is optimal if :

. s_1 of its components are equal to $(q_1+1)\xi$.
. $\mathcal{E}(a)-s_1$ of its components are equal to $q_1\xi$. (2-20)
. the remaining components are equal to zero.

2.2. In the case of $a\neq\mathcal{E}(a)$, an admissible control is optimal either if it verifies (2-20) or if it verifies :

$$\begin{cases} \cdot\ s_2 \text{ of its components are equal to } (q_2+1)\xi. \\ \cdot\ \mathcal{E}(a)+1-s_2 \text{ of its components are equal to } q_2\xi. \\ \cdot \text{ the remaining components are equal to zero.} \end{cases} \qquad (2\text{-}21)$$

Proof :

a. Let D_r be the set of admissible controls which have r strictly positive components ($1 \leq r \leq N$) and which verify $\nu_i \cdot y_i=0$, $i=0,1,\ldots,N-1$ (ν_i are the components of the control and y_i are the corresponding stock levels). We set $q=\mathcal{E}(N/r)$ and $s=N-qr$ and we denote by D_r^* the set of controls which verify :

1. $D_r^* \subset D_r$

2. $\begin{cases} N-\mathcal{E}(N/r).r \text{ of its components are equal to } (\mathcal{E}(N/r)+1)\xi. \\ (1+\mathcal{E}(N/r))r-N \text{ of its components are equal to } \mathcal{E}(N/r)\xi. \\ \text{the remaining components are equal to zero.} \end{cases}$

We first prove that, whatever $V^* \epsilon D_r^*$ may be, then

$$\mathscr{C}(y_0,V^*) = \underset{V \epsilon D_r}{\mathrm{Min}}\, \mathscr{C}(y_0,V)$$

If $V=\{\nu_0,\nu_1,\ldots,\nu_{N-1}\} \epsilon D_r$, then :

$$\begin{cases} \nu_0=n_1\xi,\ \nu_{n_1}=n_2\xi,\ldots,\nu_{n_1+n_2+\ldots+n_{r-1}}=n_r\xi. \\ \\ \nu_i=0 \text{ for } i \epsilon \{0,1,\ldots,N-1\} \text{ and } i \not\epsilon \{0,n_1,n_1+n_2,\ldots,n_1+n_2+\ldots+n_{r-1}\} \end{cases}$$

and

The cost is then :

$$\mathscr{C}(y_0,V)= rK+ aN\xi+ b\xi \sum_{i=1}^{r} \left(\sum_{j=1}^{n_i-1} j \right) \qquad (2\text{-}22)$$

assuming that $\sum_{j=1}^{n_i-1} j=0$ when $n_i=1$

The minimum of (2-22) is obtained by choosing $n_1,n_2,\ldots,n_r$ which minimize :

$$\sum_{i=1}^{r}\left(\sum_{j=1}^{n_i-1} j\right) \quad \text{with} \quad \sum_{i=1}^{r} n_i=N$$

The solution of this problem is known :

$$\begin{cases} n_i=\mathcal{E}(N/r)+1 & \text{for } N-\mathcal{E}(N/r).r \text{ of the i-index.} \\ n_i=\mathcal{E}(N/r) & \text{for the } (1+\mathcal{E}(N/r)).r-N \text{ others.} \end{cases}$$

The proof of the first part of the theorem is then finished.

b. Let us now consider $V^r\epsilon\, D_r^*$, $r\,\epsilon\,\{1,2,\ldots,N\}$.

Starting from the previous results, (2-21) becomes :

$$\frac{1}{N}\mathscr{C}(y_0,V^r)=[K-b\xi q(q+1)/2]r/N+ b\xi q+ a\xi \qquad (2\text{-}23)$$

with $q=\mathcal{E}(N/r)$

The problem consists in finding $V^{r_1}\epsilon\, D_{r_1}^*$ which verifies :

$$\mathscr{C}(y_0,V^{r_1})=\min_{r\epsilon\{1,2,\ldots,N\}}\mathscr{C}(y_0,V^r)$$

The function (2-23) is continue on $[1,N]$, linear on every interval $[N/(N-i+2),\ N/(N-i+1)]$, $i=2,3,\ldots,N$. In addition, it is easy to prove that this function is convex.

Let r_0 be the smallest value of r verifying

$$h(q)= K- b\xi q(q+1)/2 \geqq 0 \text{ (see (2-23))}$$

This value is the smallest integer greater than or equal to p. Starting from this remark, a short discussion leads to the theorem. $\square$

2.5. ECONOMICAL ORDERING QUANTITY (E.O.Q.) FOR THE MONO-PRODUCT PROBLEM

The E.O.Q. is the integer value which minimize the total cost per time unit if the horizon is infinite, i.e.:

$$w(q) = \frac{K}{q} + b\xi(q-1)/2 + a\xi \tag{2-24}$$

The relation (2-24) is obtained starting from (2-23) by replacing $\frac{r}{N}$ by $\frac{1}{q} = \frac{1}{\mathcal{E}(N/r)}$. It is the cost per time unit expressed using the mean replenishment and it is a convex function of q.

The minimal value of w(q) is obtained by setting :

$$q = \bar{q} = \sqrt{2K/(b\xi)} \tag{2-25}$$

The optimal replenishment $q^*\xi$ is then obtained as follows :

$$\left.\begin{array}{l}
\text{1. If } \bar{q} \leqq 1, \text{then } q^*=1 \text{ : each component of the optimal control is equal to } \xi. \\
\text{2. If } \bar{q} > 1, \text{ we have to consider two cases :} \\
\quad \text{2.1. If } \bar{q} = \mathcal{E}(\bar{q}),\ q^* = \bar{q} \\
\quad \text{2.2. If } \bar{q} \neq \mathcal{E}(\bar{q}), \text{ then :} \\
\qquad \begin{cases} q^* = \mathcal{E}(\bar{q}) \text{ if } w[\mathcal{E}(\bar{q})] < w[\mathcal{E}(\bar{q})+1] \\ q^* = \mathcal{E}(\bar{q})+1 \text{ if not} \end{cases}
\end{array}\right\} \tag{2-26}$$

It is easy to show that :

$$p \leqq \bar{q} < p+1 \quad \text{(see theorem (2-16))} \tag{2-27}$$

3. THE STATIONARY TWO PRODUCTS-PROBLEM WITH JOINT PRODUCTION COSTS

3.1. NOTATION

We consider the N-horizon problem in the case of two different products denoted by 1 and 2. For $j=1,2$, ξ^j is the demand of product j at time i ($i= 1,2,\ldots,N$) and v_i^j is the production level (or replenishment) concerning the product j, decided at time i and available at time i+1 ($i= 0,1,\ldots,N-1$).

We denote by y_i^j ($j= 1,2$ and $i= 0,1,\ldots,N$) the inventory level of product j on $[i,i+1)$.

The following relations are called state equations :

$$y_{i+1}^j= y_i^j +v_i^j -\xi^j, \quad j= 1,2 \quad \text{and} \quad i= 0,1,\ldots,N-1 \tag{3-1}$$

Backlogging is not allowed, then :

$$y_i^j \geqq 0 \quad \text{for } i= 0,1,\ldots,N \quad \text{and} \quad j= 1,2 \tag{3-2}$$

and we assume that :

$$y_0^1= y_0^2= 0 \tag{3-3}$$

The following notation is used :

$$V= \{v_i^j\}_{i=0,1,\ldots,N-1}^{j=1,2} = \{V^j\}^{j=1,2} = \{V_i\}_{i=0,1,\ldots,N-1}$$

V is a control and :

$$Y= \{y_i^j\}_{i=0,1,\ldots,N}^{j=1,2} = \{Y^j\}^{j=1,2} = \{Y_i\}_{i=0,1,\ldots,N}$$

is the set of inventory levels corresponding to V starting from y_0^1, y_0^2 and (3-1).

If Y verifies (3-2), V is said to be admissible.

We also consider :

1. for $i = 0,1,\ldots,N-1$ and $j=1,2$, $f_i^j(y) = b_j y (b_j \in \mathbb{R}^{*+}$ and $y \in \mathbb{R}^+)$. It is the cost of holding in stock a quantity y of the product j at time i for one period until i+1.

2. for $i=0,1,\ldots,N-1$ and $V_i=\{\nu_i^1,\nu_i^2\}$, the cost of ordering a quantity ν_i^1 of product 1 and ν_i^2 of product 2 at time i (also called production cost) is :

$$c_i(V_i) = K^{J(V_i)} + a_1\nu_i^1 + a_2\nu_i^2$$

$(a_1 \in \mathbb{R}^+,\ a_2 \in \mathbb{R}^+, \nu^1 \in \mathbb{R}^+, \nu^2 \in \mathbb{R}^+, K^{J(V_i)} \in \mathbb{R}^+$ and :

$$J(V_i) = \{ x_{\nu_i^1>0}, x_{\nu_i^2>0} \}$$

The following definition holds :

$$x_a = \begin{cases} 1 \text{ if the condition a is true} \\ 0 \text{ if not} \end{cases}$$

We assume that :

$$\begin{cases} 0 \leq \mathrm{Max}[K^{\{1,0\}}, K^{\{0,1\}}] \leq K^{\{1,1\}} \leq K^{\{1,0\}} + K^{\{0,1\}} \\ K^{\{0,0\}} = 0 \end{cases} \tag{3-4}$$

The total cost corresponding to V is then :

$$\mathcal{C}(y_0^1, y_0^2, V) = \sum_{i=0}^{N-1} [K^{J(V_i)} + a_1\nu_i^1 + a_2\nu_i^2 + b_1 y_i^1 + b_2 y_i^2] \tag{3-5}$$

If D is the set of admissible controls, V* is said to be optimal if :

$$\mathcal{C}(y_0^1, y_0^2, V^*) = \underset{V\in D}{\mathrm{Min}}\ (y_0^1, y_0^2, V) \tag{3-6}$$

This problem will be denoted $P_2(N)$ in the following. $P_2(\infty)$ is the infinite horizon problem.

3.2. AN E.O.Q. ALGORITHM FOR THE TWO-PRODUCTS PROBLEM

We denote by $G_k(N)(k=1, 2)$ the mono-product problem with the initial stock level $y_0^k=0$, a demand ξ^k at each time i (i=1,2,...,N), the inventory cost function f_i^k and the ordering (or production) cost function :

$$z_i^k(\nu)= K_k+ a_k\nu, \quad \nu \geq 0 \tag{3-7}$$

where :

$$K_k= K^{(1,0)} \text{ if } k=1 \text{ and } K_k= K^{(0,1)} \text{ if } k=2$$

$G_k(N)$ becomes $G_k(\infty)$ when the horizon is infinite.

We first give some properties of the two-products infinite horizon problem. These properties will be used in the following algorithm.

Theorem (3-1)

At least one optimal control of $P_2(\infty)$ is periodic.

Proof :

Let us denote by $V=\{V^1,V^2\}$ an optimal control of $P_2(\infty)$.

a) We first prove that there exist $0= n_1< n_2< \ldots<\ldots$ which verify :

$$\nu^1_{n_k} \cdot \nu^2_{n_k} > 0 \text{ for } k= 1, 2,\ldots \tag{3-8}$$

Suppose that (3-8) does not hold. For k=1,2, V^k would be the optimal control of $G_k(\infty)$, and we know that it is periodic. This hypothesis leads to (3-8) for k=r where r is the least common multiple of the two periods : this contradicts the fact that (3-8) does not hold.

b) Because V is optimal, its restriction to any period $[n_k, n_{k+1}]$, k= 1, 2,...., leads to the same cost per time unit. Consequently, the concatenation of the restriction of V to any $[n_k,n_{k+1}]$ periodic time with itself is optimal and periodic. □

Theorem (3-2)

Let us denote by T the period of the optimal control of $P_2(\infty)$. Let V^k (k=1,2) be the optimal control of $G_k(T)$. Then :

$V = (V^1, V^2)$ is an optimal control for $P_2(T)$.

Proof :

V^k being optimal for $G_k(T)$, it is also optimal for the problem $G_k^*(T)$ obtained starting from $G_k(T)$ by replacing the production cost function $z_0^k(\nu)$ by :

$$z_0^{*k}(\nu) = K^{(1,1)} - K_{3-k} + a_k\nu \quad \text{(see 3-7)}$$

This item is a consequence of the theorem (2-1), taking into account the fact that $K^{(1,1)} - K_{3-k} \leq K_k$ (see (3-4)). Finally :

V_1 is an optimal control for $G_1^*(T)$ and V_2 is an optimal control for $G_2(T)$.

Then $V = (V^1, V^2)$ is optimal for $P_2(T)$. □

Theorem (3-3)

Let $q_k^* \xi^k$ be the strictly positive components of $V^k(\infty)$, optimal control of $G_k(\infty)$ (k= 1, 2).

Let $\mathscr{T}$ be the least common multiple of q_1^* and q_2^*. $\mathscr{T}$ is an upper bound of the period of the optimal control of $P_2(\infty)$.

Proof :

$$\mathscr{T} = n_k q_k^*, \quad k = 1,2.$$

Whatever $N > \mathscr{T}$ may be, $V^k(N)$, optimal control of $G_k(N)$, contains more than n_k components equal to $q_k^* \xi^k$: it is a consequence of (2-27) and theorem 2-3.

For $k= 1,2$, we choose the n_k first strictly positive components of $V^k(N)$ equal to $q_k^* \xi^k$. It leads to $v^k_{\mathscr{T}}(N)>0$ and $v^1_{\mathscr{T}}(N).v^2_{\mathscr{T}}(N) > 0$. Then $y^1_{\mathscr{T}}(N)= y^2_{\mathscr{T}}(N)= 0$, where $y^1_{\mathscr{T}}(N)$ and $y^2_{\mathscr{T}}(N)$ are the inventory levels of products 1 and 2 on $[\mathscr{T},\mathscr{T}+1)$, and this item is true whatever $N > \mathscr{T}$ may be. The theorem is proved.

□

The following algorithm is based on the previous results.

1. <u>For $k= 1,2$</u> , compute (see theorem 3-3 and paragraph 2.5):
   - 1.1. q_k^*
   - 1.2. $\mathscr{C}^k(\infty)$, optimal cost per time unit of $G_k(\infty)$
2. Compute $\mathscr{T}$, least common multiple of q_1^* and q_2^*.
3. Set :
   - 3.1. $h=\mathscr{T}$
   - 3.2. $U=\mathscr{C}^1(\infty)+\mathscr{C}^2(\infty)$, upper bound of the optimal cost per time unit of $P_2(\infty)$ (see theorem (2-1) and its proof)
4. <u>For $m= 1,2,3,\ldots,$</u> (see theorems 3-1 , 3-2 and 3-3):
   - 4.1. <u>For $k= 1,2$</u> , compute (see theorem 2-3):
     - 4.1.1. $\mathscr{C}^k(m)$, optimal cost per time unit of $G_k(m)$
     - 4.1.2. $V^k(m)$, optimal control of $G_k(m)$
   - 4.2. Compute :
     - $a=\mathscr{C}^1(m)+\mathscr{C}^2(m)-\frac{1}{m}\left[K^{(1,0)}+K^{(0,1)}-K^{(1,1)}\right]$
   - 4.3. <u>If $a\leq U$ and $v_i^1(m).v_i^2(m)=0$ for $i= 1,2,\ldots, m-1$</u> :
     - 4.3.1. $U= a$
     - 4.3.2. $h=m$
     - 4.3.3. $V= (V^1(m),V^2(m))$

Finally, U is the optimal cost per time unit of the two products problem $P_2(\infty)$, h is the period of the optimal control and V is the restriction to $[0,h]$ of the optimal control of $P_2(\infty)$.

4. AN EXAMPLE

We use the notation of the paragraph 3.1.

We choose :

$$\xi^1 = \xi^2 = 1$$

$$a_1 = a_2 = 0$$

$$K^{(1,0)} = 15,\ K^{(0,1)} = 4{,}5\ ,\ K^{(1,1)} = 15$$

$$b_1 = b_2 = 0$$

Using (2-25), we obtain the following results :

1. the period of the optimal control of $G_1(\infty)$ is 5 or 6.
2. the period of the optimal control of $G_2(\infty)$ is 3.

We then have to compute the optimal controls of $G_1(N)$ and $G_2(N)$ for N= 1,2,3,4,5,6, to deduce the optimal control of $P_2(N)$ for every value of N and to retain the optimal control corresponding to the smallest cost per time unit.

We summarize these results in the following table :

N	Optimal cost of $G_1(N)$	Optimal cost of $G_2(N)$	Optimal cost per time unit of $P_2(N)$
1	15	4,5	15+4,5-4,5 = 15
2	16	5,5	(16+5,5-4,5)/2 = 8,5
3	18	7,5	(18+7,5-4,5)/3 = 7
4	21	10,5	(21+10,5-4,5)/4 = 6,75
5	25	13	(25+13-4,5)/5 = 6,7
6	30	15	(30+15-4,5)/6 = 6,75

The problem which leads to the smallest optimal cost per time unit is then $P_2(5)$ and the optimal control of this problem is $V = \{V^1(5), V^2(5)\}$, where $V^k(5)$ is the optimal control for $G_k(5)$ $(k=1,2)$.

We obtain :

$$V^1(5) = (5,0,0,0,0)$$

and

$$V^2(5) = (3,0,0,2,0)$$

Finally, the optimal policy for the infinite horizon problem consists in ordering product 1 as shown by $V^1(5)$ and product 2 as shown by $V^2(5)$ on every five time units period.

5. CONCLUSION

The above algorithm is based on the following properties :

1. if T is the period of the infinite horizon two products problem optimal cost, then its restriction to the T-horizon is optimal for the T-horizon problem.
2. an upper bound of T exists, and this upper bound is easy to obtain. Let us denote by $\mathcal{T}$ this upper bound.
3. we only have to compute two optimal controls of mono-product N-horizon problems for $N = 1,2,\ldots,\mathcal{T}$ in order to obtain the optimal control of the T-horizon problem.

Properties 1 and 2 hold for the M-product problem, whatever $M \geqq 2$ may be. Unfortunately, property 3 is only true for the two-products problem. It is then possible to obtain an algorithm similar to the previous one whatever $M \geqq 2$ may be, but it needs the computation of the optimal control of the M-products N-horizon problem for $N = 1,2,\ldots,\mathcal{T}$. The exact solution of this problem is known in the general case (see [10]), but the amount of computation is proportional to N^{M+1}.

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Imprimé en France

par

l'Institut National de Recherche en Informatique et en Automatique